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1990 J. Phys. A: Math. Gen. 23 L609

(http://iopscience.iop.org/0305-4470/23/12/005)

J. Phys. A: Math. Gen. 23 (1990) L609-L612. Printed in the UK

LETTER TO THE EDITOR

Eigenfunction equations in Sato theory

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Received 30 March 1990

Abstract. Soliton eigenfunction equations within the framework of the Sato (Σ-function) method are considered. The universal eigenfunction equations of the first and second levels are constructed.

Nonlinear partial differential equations integrable by the inverse spectral transform (IST) method form a wide class of soliton equations which possess many remarkable properties [1-3]. Soliton equations can be described in different ways. The Sato approach, based on the use of the infinite-dimensional Grassmann manifold, is one of the most beautiful ways [4] (see also [5-7]). The so-called Σ-function is a key notion of this method.

Recently it has been shown [8-10] that the eigenfunctions of soliton eigenvalue problems also obey nonlinear partial differential equations and that they are also integrable by the IST method. Soliton eigenfunction equations have many interesting properties [8-10]. They are important ingredients of the theory of integrable equations.

In the present letter we consider the soliton eigenfunction equations within the frame of the Sato theory. We construct the universal nonlinear equation for the eigenfunction ψ of the Kadomtsev-Petviashvili (KP) hierarchy. In such a formulation the Σ-function is the KP eigenfunction of the second level and the nonlinear equation for the Σ-function is nothing but the universal KP eigenfunction equation of the second level.

The auxiliary linear problem for the KP equation $(u_t-\frac{1}{4}u_{xxx}-\frac{3}{4}uu_x)_x-\frac{3}{4}u_{yy}=0$ is of the form [1, 2]

$$\begin{aligned} &\psi_y-\psi_{xx}-2u\psi=0\\ &\psi_t-\psi_{xxx}-3u\psi_x-\tfrac{3}{2}u_x\psi-\tfrac{3}{2}(\partial_x^{-1}u_y)\psi=0. \end{aligned} \tag{1}$$

Elimination of the potential u from (1) gives rise to the KP eigenfunction equation [8-10]

$$(\psi_t\psi^{-1})_x-\tfrac{1}{4}(\psi_x\psi^{-1})_{xxx}+\tfrac{1}{2}((\psi_x\psi^{-1})^3)_x-\tfrac{3}{4}(\psi_y\psi^{-1})_y-\tfrac{3}{2}(\psi_x\psi^{-1})_x\psi_y\psi^{-1}=0. \tag{2}$$

Equation (2) is integrable by the IST method with the help of the linear problem [8-10]

$$\begin{aligned} &\psi\varphi_y-\psi\varphi_{xx}-2\psi_x\varphi_x=0\\ &\psi\varphi_t-\psi\varphi_{xxx}-3\psi_x\varphi_{xx}-\tfrac{3}{2}(\psi_{xx}+\psi_y)\varphi_x=0. \end{aligned} \tag{3}$$

In (3) ψ plays the role of potential while φ is the eigenfunction. Eliminating now ψ from (3), one arrives at the KP eigenfunction equation of the second level, i.e. the eigenfunction equation for the eigenfunction equation [10]. If one continues this

0305-4470/90/120609+04$03.50

process then one can show [10] that the eigenfunction equations of the third and all higher levels coincide with the second-level equation. So the whole vertical hierarchy of the KP eigenfunction equations (i.e. the family of eigenfunction equations for eigenfunction equations etc.) contains only the two different members.

In the same manner one can consider all the higher KP equations (horizontal KP hierarchy) and construct the corresponding eigenfunction equations of the first and second levels. The properties of the vertical hierarchies for all the higher KP equations are the same.

Within the framework of the Sato theory [4-7] the whole horizontal KP hierarchy is described by the system

$$\frac{\partial L}{\partial t_n}=[L, B_n] \qquad n=1,2,3,\ldots \tag{4}$$

where L is the pseudodifferential operator $L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\ldots$, $\partial\equiv\partial/\partial x$, the coefficients u_k depend on the infinite set of variables $t_1=x$, t_2, $t_3,\ldots$ and $B_n=(L^n)_+$ is the pure differential part of the operator L^n. The system (4) is the infinite system of equations of the form

$$\begin{aligned}
&\frac{\partial u_1}{\partial t_2}=u_{1xx}+2u_{2x}\\
&\frac{\partial u_2}{\partial t_2}=u_{2xx}+2u_{3x}+2u_1u_{1x}\\
&\vdots\\
&\frac{\partial u_1}{\partial t_3}=u_{1xxx}+3u_{2xx}+3u_{3x}+6u_1u_{1x}.\\
&\vdots
\end{aligned} \tag{5}$$

Elimination of u_2, $u_3,\ldots$ from (5) gives rise to the usual KP hierarchy for u_1 ($t_2=y$).

The nonlinear system (4) is the compatibility condition for the linear system [4-7]

$$L\psi=\lambda\psi \tag{6}$$

$$\frac{\partial\psi}{\partial t_n}=B_n\psi \qquad n=1,2,3,\ldots. \tag{7}$$

From (7) it follows also that

$$\frac{\partial B_m}{\partial t_n}-\frac{\partial B_n}{\partial t_m}-[B_n, B_m]=0 \qquad n, m=1,2,3,\ldots. \tag{8}$$

The first two equations (7) are of the form

$$\begin{aligned}
&\frac{\partial\psi}{\partial t_2}=(\partial^2+2u_1)\psi\\
&\frac{\partial\psi}{\partial t_3}=(\partial^3+3u_1\partial+3u_2+3u_{1x})\psi
\end{aligned} \tag{9}$$

and so on.

Now let us apply to the system (6), (7) the approach proposed in references [8-10], i.e. eliminate the potentials u_1, u_2, $u_3,\ldots$ from (6), (7). This can easily be done by the use of (7). Indeed, all the operators $B_n = (L^n)_+$ have a triangular structure and contain the function u_n linearly. As a result, from (7) one has

$$u_n = F_n(\psi) \qquad n = 1, 2, 3, \ldots \tag{10}$$

where F_n are certain explicit functions on ψ, ψ_{t_1}, $\psi_{t_2}, \ldots$. For example,

$$\begin{aligned} u_1 &= F_1(\psi) = \frac{\psi_{t_2} - \psi_{xx}}{2\psi} \\ u_2 &= F_2(\psi) = \frac{2\psi_{t_3} - 3\psi_{xt_2} - 3\psi_{xxx}}{6\psi} \end{aligned} \tag{11}$$

and so on.

Substituting the expressions (10) into (6), we obtain the equation

$$\psi_x + \sum_{n=1}^{\infty} F_n(\psi)\partial^{-n}\psi = \lambda\psi. \tag{12}$$

This is the equation for the common KP hierarchy eigenfunction ψ in the Sato approach. We emphasise that in Sato theory we have a single eigenfunction equation in contrast to the horizontal hierarchy of eigenfunction equations discussed in [10]!

The functions $F_n(\psi)$ in (12) depend on ψ, its derivatives with respect to all the variables t_1, t_2, $t_3, \ldots$ and higher-order and cross derivatives. But equations (8) imply that all the derivatives ψ_{t_n} at $n \geqslant 3$ can be expressed via ψ, ψ_x and ψ_{t_2}. Therefore, equation (12) can also be rewritten in the form

$$\psi_x + \sum_{n=1}^{\infty} \tilde{F}_n(\psi)\partial^{-n}\psi = \lambda\psi \tag{13}$$

where $\tilde{F}_n$ are the functions only on ψ, ψ_x, ψ_{t_2} and cross derivatives with respect to x and t_2.

So in the Sato theory the eigenfunction ψ obeys the single nonlinear, non-local, effectively two-dimensional (x, t_2) equation (13).

Universality is the main feature of the eigenfunction equation (13) (or (12)). It contains the complete information about the whole Sato KP hierarchy.

One can consider the adjoint common KP eigenfunction ψ^* instead of ψ. This function obeys the adjoint system (6), (7). Elimination of the potentials u_n gives rise to, in this case, the adjoint eigenfunction equation

$$-\psi_x^* + \sum_{n=1}^{\infty} (-\partial)^{-n} F_n^*(\psi^*)\psi^* = \lambda\psi^*. \tag{14}$$

Equation (12) and equation (14) give the two forms of the eigenfunction equation. In the Sato theory there also exists a very simple form of the eigenfunction equation which contains both ψ and ψ^*. This is nothing but the well known bilinear identity [5-7]

$$\int_{\Gamma_\infty} \mathrm{d}\lambda\; \psi(t, \lambda)\psi^*(t', \lambda) = 0. \tag{15}$$

One can prove (see e.g. [5-7]) that if the functions ψ and ψ^* are of the form

$$\psi(t, \lambda) = \sum_{n=0}^{\infty} \psi_n(t)\lambda^{-n} \exp \xi(t, \lambda)$$

$$\psi^*(t, \lambda) = \sum_{n=0}^{\infty} \psi_n^*(t)\lambda^{-n} \exp(-\xi(t, \lambda))$$

where $\psi_0=\psi_0^*=1$ and $\xi(t,\lambda)=\Sigma_{n=1}^{\infty} t_n\lambda^n$ and if they obey equation (15), then ψ is the common eigenfunction for the KP hierarchy and ψ^* is the adjoint eigenfunction. This result implies that the bilinear-bilocal equation (15) is equivalent both to equation (12) and equation (14).

Note that one can rewrite equations (12) and (14) in the terms of the functions χ and χ^* introduced by $\psi=\chi\exp\xi(t,\lambda)$ and $\psi^*=\chi^*\exp(-\xi(t,\lambda))$. The corresponding equations contain the complicated dependences on λ.

Equations (12) and (14) are the eigenfunction equations of the first level. What about the eigenfunction and the eigenfunction equation of the second level?

The eigenfunction of the second level is, in fact, a well known object in Sato theory. This is nothing but the Σ-function. Indeed, the function is introduced via the relation [4–7]

$$\Sigma\left(t_1-\frac{1}{\lambda}, t_2-\frac{1}{2\lambda^2}, \ldots\right)=\psi(t,\lambda)\,\mathrm{e}^{-\xi(\lambda,t)}\Sigma(t). \qquad (16)$$

Here $\psi(t,\lambda)$ plays the role of potential and Σ is the corresponding eigenfunction. Equation (16) is the analogue of the linear system (3) in the Sato theory.

Substituting the expression for ψ via Σ into (12), one gets the eigenfunction equation of the second level in the Sato theory. Another, more compact form of this equation follows from (15) and it is nothing but the known bilinear-bilocal equation for the Σ-function [4–7]

$$\int_{\Gamma_\infty} \mathrm{d}\lambda\, \Sigma\left(t_1-\frac{1}{\lambda}, t_2-\frac{1}{2\lambda^2}, \ldots\right)\Phi\left(t_1'+\frac{1}{\lambda}, t_2'+\frac{1}{2\lambda^2}, \ldots\right)\mathrm{e}^{\xi(t-t',\lambda)}=0. \qquad (17)$$

Similar to the eigenfunction equation (12) equation (17) is the universal one. The Σ-function is the fundamental object and it contains complete information about the KP hierarchy and its properties [4–7].

In a similar manner one can consider not only the AKP hierarchy, discussed above, but also the BKP, DKP and SKP hierarchies [5, 6]. The (1+1)-dimensional case can be treated via the so-called n-reductions (see e.g. [5–7]).

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